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Fractional Brownian Motions and Enhanced Diffusion in a Unidirectional Wave-Like Turbulence

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We study transport in random undirectional wave-like velocity fields with nonlinear dispersion relations. For this simple model, we have several interesting findings: (1) In the absence of molecular diffusion the entire family of fractional Brownian motions (FBMs), persistent or anti-persistent, can arise in the scaling limit. (2) The infrared cutoff may alter the scaling limit depending on whether the cutoff exceeds certain critical value or not. (3) Small, but nonzero, molecular diffusion can drastically change the scaling limit. As a result, some regimes stay intact; some (persistent) FBM regimes become non-Gaussian and some other FBM regimes become Brownian motions with enhanced diffusion coefficients. Moreover, in the particular regime where the scaling limit is a Brownian motion in the *absence* of molecular diffusion, the vanishing molecular diffusion limit of the enhanced diffusion coefficient is strictly larger than the diffusion coefficient with zero molecular diffusion. This is the first such example that we are aware of to demonstrate rigorously a nonperturbative effect of vanishing molecular diffusion on turbulent diffusion coefficient.

KEY WORDS: Fractional Brownian motions; wave turbulence; convection enhanced diffusion.

1. INTRODUCTION

Turbulent transport of passive tracer particles is an important problem in fluid dynamics and applied mathematics because of its potential applications to containment transport in the environment (ground water, oceans or the atmosphere). At a theoretical level, it provides a simple mathematical model which can exhibit rich varieties of structures and phenomena.

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Let $\mathbf{v}(t, \mathbf{x})$ be a given velocity field and let $\rho(t, \mathbf{x})$ be the concentration of particles at time t, satisfying the advection-diffusion equation

$$\partial\rho/\partial t + \nabla \cdot [\mathbf{V}(t, \mathbf{x})\, \rho] = D\Delta\rho, \qquad \rho(t=0, \mathbf{x}) = \rho_0(\mathbf{x}) \tag{1}$$

where $D \geqslant 0$ is the molecular diffusivity. The concentration $\rho(t, \mathbf{x})$ can be solved for from Eq. (1) as

$$\rho(t, \mathbf{x}) = \int G(t, \mathbf{x}, \mathbf{y})\, \rho_0(\mathbf{y})\, d\mathbf{y}$$

where $G(t, \mathbf{x}, \mathbf{y})$ is the transition probability density of finding, at time t, the particle at point $\mathbf{x}$, given that the particle is released at $\mathbf{y}$ at time 0. We study the particle's sample path $\mathbf{x}(t)$ which satisfies the Ito stochastic differential equation

$$d\mathbf{x}(t) = \mathbf{V}(t, \mathbf{x}(t))\, dt + \sqrt{2D}\, d\mathbf{w}(t), \qquad \mathbf{x}(0) = \mathbf{x}_0 \tag{2}$$

where $\mathbf{w}(t)$ is the standard Brownian motion. The velocity field $\mathbf{V}(t, \mathbf{x})$ is assumed to be a time stationary, space-homogeneous random vector field. The primary object of investigation is the limiting law, as $\varepsilon \to 0$, of the scaling transformation

$$\varepsilon \mathbf{x}(t/\varepsilon^{2\delta}) \tag{3}$$

for some appropriate $\delta > 0$.

In the previous papers,[10, 11] we considered turbulent transport in an *isotropic* Ornstein–Uhlenbeck velocity field $\mathbf{V}$ with the correlation function $R_{ij}(t, \mathbf{x})$ given by

$$R_{ij}(\mathbf{x}, t) = \int_{R^d} r(|\mathbf{k}|^{2\beta}\, t) \cos(\mathbf{k} \cdot \mathbf{x})\, \mathscr{E}(\mathbf{k})(\delta_{ij} - k_i k_j\, |\mathbf{k}|^{-2})\, |\mathbf{k}|^{1-d}\, d\mathbf{k}, \qquad d \geqslant 2 \tag{4}$$

where

$$r(|\mathbf{k}|^{2\beta}\, t) = \exp(-|\mathbf{k}|^{2\beta}\, t), \qquad \beta > 0 \tag{5}$$

is the time correlation function and

$$\mathscr{E}(\mathbf{k}) = a(|\mathbf{k}|)\, |\mathbf{k}|^{1-2\alpha} \tag{6}$$

is the power spectrum of the velocity field. A nonnegative function $a(|\mathbf{k}|)$ appearing in the power spectrum above represents cutoffs (infrared or ultraviolet) necessary for $\hat{R}_{ij}(\mathbf{k}, t)$ to be integrable in $\mathbf{k}$. In particular, an ultraviolet cutoff is needed for $\alpha < 1$. That is, for $\alpha < 1$, $a(|\mathbf{k}|)$ has a compact support in wave numbers and is supported, say in $[0, K_0]$, $K_0 < \infty$. When there is no infrared cutoff, we have $a(0) > 0$.

In the weak coupling limit (see refs. 10 and 11), we showed that the displacement $\mathbf{x}(t)$ converges weakly to fractional Brownian motions (FBMs) in the region defined by $\alpha + \beta > 1$ as a result of long-range correlation. Interestingly, these fractional Brownian motions are invariably *persistent* in the sense that their Hurst exponent H ranges between 1/2 and 1. $H = 1/2$ corresponds to a Brownian motion (BM) limit; $H = 1$ corresponds to a regular motion. Anti-persistent FBMs $(0 \leqslant H < 1/2)$ do not occur in the scaling limit for these Ornstein–Uhlenbeck velocity fields. In contrast to Brownian motion, which is a Gaussian Markov process, FBMs are Gaussian but not Markovian. The limit density $\bar{\rho}(t, \mathbf{x})$ satisfies an integro-differential equation which, in the persistent case $H \in (1/2, 1)$, takes the form

$$\frac{\partial}{\partial t} \bar{\rho}(t, \mathbf{x}) = \Gamma^{-1}(2H-1) \int_0^t (t-s)^{2H-2} D^* \Delta \bar{\rho}(s, \mathbf{x}) \, ds, \qquad \bar{\rho}(0, \mathbf{x}) = \rho_0(\mathbf{x}) \tag{7}$$

and, in the antipersistent case $H \in (0, 1/2)$, takes the form

$$\Gamma^{-1}(-2H+1) \int_0^t (t-s)^{-2H} \frac{\partial}{\partial s} \bar{\rho}(s, \mathbf{x}) \, ds = D^* \Delta \bar{\rho}(t, \mathbf{x}), \qquad \bar{\rho}(0, \mathbf{x}) = \rho_0(\mathbf{x}) \tag{8}$$

Here $\Gamma(h)$ is the Gamma function. Note that the integral kernels in both equations are integrable functions for the respective range of H. In contrast to (1), (8) and (7) are *nonlocal in time*.

The weak coupling limit weakens and discounts the *spatial* variation of the velocity field, so it is simpler to study than the original problem. As a result, the results are insensitive to the compressibility of the velocity field (cf. ref. 19). Nevertheless, besides interesting on its own right, the results for the weak coupling limit are believed to hold for the usual scaling limit for a substantial range of parameters α, β of the above Ornstein–Uhlenbeck flows.

Decaying temporal correlation, such as (5), is more typical of hydrodynamic turbulence, whereas oscillating temporal correlation is often present in wave turbulence (ref. 28). In the present paper, we consider Gaussian velocity fields with an oscillating time correlation function

$$r(|\mathbf{k}|^{2\beta} t) = \cos(\Omega(|\mathbf{k}|) \, t) \tag{9}$$

with a non-linear dispersion relation

$$\Omega(|\mathbf{k}|) = |\mathbf{k}|^{2\beta}, \qquad \beta \geqslant 0 \tag{10}$$

Such flows are a superposition of random waves propagating at different phase speeds determined by the non-linear dispersion relation (10) and they arises in many practical situations related to wave turbulence (see, e.g., ref. 28). When $2\beta = 1$, there is no dispersion; when $2\beta > 1$, short, waves travel faster than long waves; when $2\beta < 1$, long waves travel faster. $\beta > 0$ because typically shorter waves have higher frequencies.

Since the transport problem for such flows is considerably more difficult, to study than that for the Ornstein–Uhlenbeck flows, a good starting point is to consider the *unidirectional flows*

$$\mathbf{V}(t, \mathbf{x}) = (0, V(t, x)), \qquad \mathbf{x} = (x, y) \in R^2$$

with the same power spectrum

$$\mathscr{E}(k) = a(k)\, |k|^{2\alpha - 1}, \qquad a(0) > 0 \qquad \alpha < 1 \tag{11}$$

In other words, the two-point co-variance function of the Gaussian velocity field $V(t, x)$ is given by

$$R(t, x) = \mathbf{E}[\, V(t, x)\, V(0, 0)] = \int_R \cos(kx) \cos(t\, |k|^{2\beta})\, \frac{a(|k|)}{|k|^{2\alpha - 1}}\, dk \tag{12}$$

The triplet of its underlying probability space is denoted by $(\Omega, \mathscr{V}, P)$. The equation of motion (2) is then simplified to

$$\begin{cases} dx(t) = \sqrt{2D}\, dw_1(t), & D \geqslant 0 \\ dy(t) = V(t, x(t))\, dt + \sqrt{2D}\, dw_2(t) \\ x(0) = 0, & y(0) = 0 \end{cases} \tag{13}$$

Here $(w_1(t), w_2(t))$, $t \geqslant 0$ is a certain two dimensional standard Brownian motion given over the probability space $(\Sigma, \mathscr{W}, Q)$. The corresponding mathematical expectation shall be denoted by $\mathbf{M}$.

Our objective is to obtain the complete information of the scaling limit, in particular, to characterize precisely the fractional Brownian motion limit, in contrast to those arising in the Ornstein–Uhlenbeck flows, and to study rigorously the subtle effect, of (vanishing) molecular diffusion, in particular the phenomenon of convection-enhanced diffusion (see, for example, refs. 25, 6, 7, 18, 20, 12, 13, 23, 22, 5 and references therein).

As remarked before, all but one regime in the unidirectional case are fully expected to carry over to the *weak coupling* limit in the isotropic case with (9) and (6) (see more discussion on this below). More differences are expected to occur in the isotropic case for the original scaling limit (3). The techniques for the isotropic case, however, are different and will be presented elsewhere. The unidirectional case for the Ornstein–Uhlenbeck flows with an additional infrared cutoff has been studied thoroughly in refs. 3 and 29. The differences between this case and the isotropic case for the original scaling limit (3) can be found in ref. 8.

System (13) can be solved explicitly and we obtain then that $x(t) = \sqrt{2D}\, w_1(t)$ and

$$y(t) = \int_0^t V(s, \sqrt{2D}\, w_1(s))\, ds + \sqrt{2D}\, w_2(t) \tag{14}$$

We ask: what is the appropriate choice of the parameter δ (depending on α, β) in the scaling transformation

$$y_{\varepsilon, \delta}(t) := \varepsilon y(t/\varepsilon^{2\delta}), \qquad \delta > 0 \tag{15}$$

such that the processes $y_{\varepsilon, \delta}(t)$ are *weakly convergent* as stochastic processes with continuous trajectories, as $\varepsilon \downarrow 0$? And what is the limit probability law, if exists? It turns out the limit process is a fractional Brownian motion uniquely characterized by its Hurst exponent $H \in [0, 1]$ which is normally related to δ by

$$H = 1/(2\delta) \tag{16}$$

a dimensionally correct equation.

We summarize the results in several diagrams (Figs. 1, 2 and 3) and briefly discuss them as follows.

Figure 1 corresponds to the case of $D = 0$ (see Section 2 for details). We show that the whole family of FBMs (persistent or anti-persistent) can be the scaling limit of turbulent transport. The persistent FBMs for $\alpha + \beta > 1$ are due to long-range correlation of the velocity field and, consequently, the transport does not feel the oscillation in the time correlation function of the velocity, so they are the same as those arising in Ornstein–Uhlenbeck velocity fields (isotropic or not).[11] The anti-persistent FBMs for $\alpha + \beta > 1$, absent in Ornstein–Uhlenbeck velocity fields, are due to the oscillation in the time correlation function (9), which tends to slow down the transport. In particular, for $\alpha + 2\beta < 1$, the limit is a bounded motion corresponding to $H = 0$.

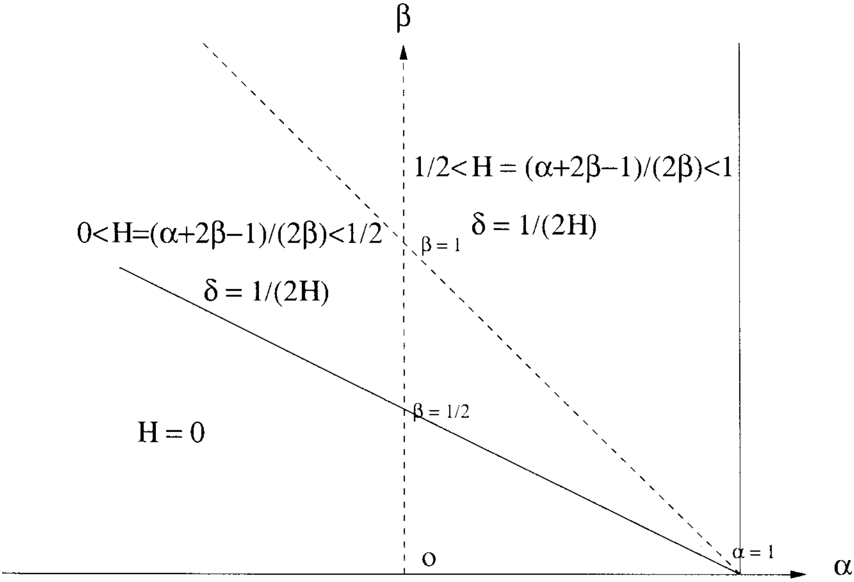

Fig. 1. The case with zero molecular diffusion, $D = 0$.

The FBM scaling law is non-Markovian and may be altered if an additional infrared cutoff exceeds certain critical power of the scaling factor ε. We discuss this effect of infrared cutoff briefly in Section 3.

Next, we discuss the effect of molecular diffusion. Figure 2 corresponds to this case (see Section 4). The effect of molecular diffusion is significant: it wipes out the anti-persistent FBM regime completely as well as part of the persistent FBM regime; it also changes the functional form of the Hurst exponent in another part ($0 < \alpha < 1, \beta > 1$) of the persistent FBM regime (Note that $(\alpha + 1)/2 < (\alpha + 2\beta - 1)/(2\beta)$ in this region). It should be noted, however, one part of the the BM regime defined by $\alpha < 0$ is part of the general homogenization theorem which is valid for any (isotropic or not, time dependent or not) space-homogeneous flows possessing space-homogeneous stream functions;[9] whereas the other part of the BM regime (defined by $\alpha + \beta < 1$) is also valid for the Ornstein–Uhlenbeck velocity fields (isotropic or not).[10] In the regime defined by $0 < \alpha < 1, \beta > 1$, the scaling limit is not a pure FBM, but rather a *composite FBM* resulting from the additional randomness of molecular diffusion (see Section 4). The composite FBM is neither Markovian nor Gaussian, thus indicative of some intermittency in transport. It is a scenario of intermittency in which the particle constantly switches among different states of motion (in this case, FBMs with different unit-time variances). Although this scenario in

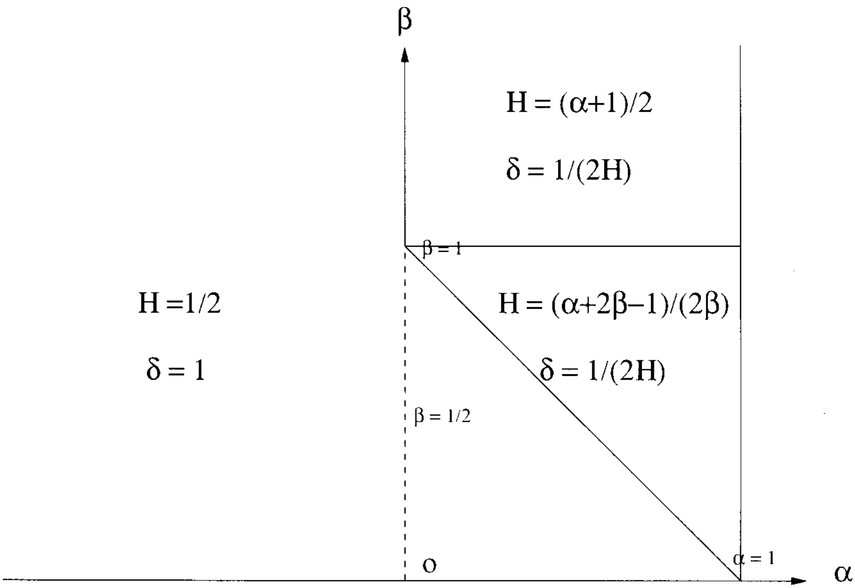

Fig. 2. The case with nonzero molecular diffusion, $D > 0$.

our case is due to the *anisotropy* of the shear flow, random mixture of Gaussian distributions as a result of molecular diffusion is a general theme of passive scalar intermittency (see ref. 21).

Finally, we consider the singular limit of vanishing molecular diffusion. Figure 3 summarizes the asymptotics as $D \to 0$. The focus is on the BM regime ($\alpha + \beta < 1$ or $\alpha < 0$) of Fig. 2. An important feature is the presence of a convection-enhanced diffusion with the effective diffusivity D^* much larger than the molecular diffusivity D. Specifically, the effective diffusivity D^* is related to D via a power-law

$$D^* \sim D^p, \qquad p = (1 - \alpha - \beta)/(\beta - 1), \qquad \text{as} \quad D \to 0 \tag{17}$$

for $\alpha + 2\beta > 2$, $\alpha < 0$. In this regime, the range of the power p is $[-1, 1]$, the fullest possible in any case (see refs. 18, 12, 13 and the references therein). p lies between 0 and 1 in the region that overlaps with the anti-persistent FBM regime of Fig. 1; whereas p lies between -1 and 0 in the region that overlaps with the persistent FBM regime of Fig. 1. That means that antipersistent as well persistent FBMs regimes in Fig. 1 can enhance turbulent diffusion.

To further understand the effect of vanishing molecular diffusion, it is particularly instructive to compare the turbulent diffusivities for the two

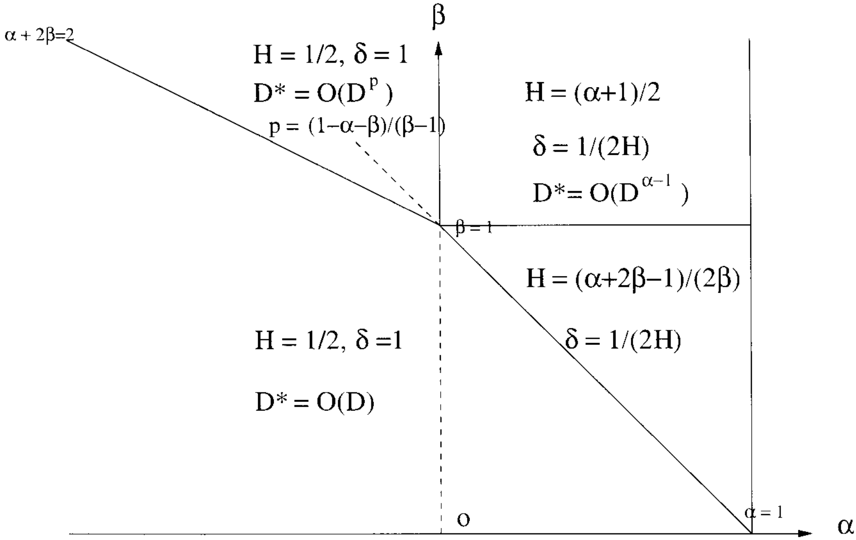

Fig. 3. The limit of vanishing molecular diffusion, $D \to 0$.

cases $D=0$ and $D \to 0$ for $\alpha+\beta=1$ where the exponent p in (17) is zero (cf. (28), (68)):

$$D^* = 2a(0) \int_0^\infty \frac{1-\cos(k^{2\beta})}{k^{2\alpha+4\beta-1}}\, dk, \qquad \text{for} \quad D=0 \tag{18}$$

$$D^* \sim 2a(0) \int_0^\infty \frac{dk}{k^{2\alpha-3}(k^4+k^{4\beta})}, \qquad \text{for} \quad D \to 0 \tag{19}$$

Both depend on the square intensity of the zero mode in the power spectrum of the velocity field, so they may be sensitive to ail additional infrared cutoff. The normalized effective diffusivities $D^*/a(0)$ are plotted in Fig. 4. Both the effective diffusivities corresponding to Eq. (19) and Eq. (18) decrease as a decreases, but the former decreases much faster than the latter. Several remarks are in order.

For the sake of discussion, let us write the dependence on D explicitly and denote by $D^*(D)$ the effective diffusivity when the molecular diffusion is D. Destructive interference in the sense that $D^*(D) < D^*(0)$ for small $D > 0$ has been predicted in ref. 25 on the basis that the presence of molecular diffusion tends to decrease the Lagrangian correlation in velocity (see also ref. 7). As rightly pointed out in refs. 22 and 5 while this may be true for flows with strictly decaying time correlation functions such as (5) it may be false if the time correlation function is oscillatory. What is striking

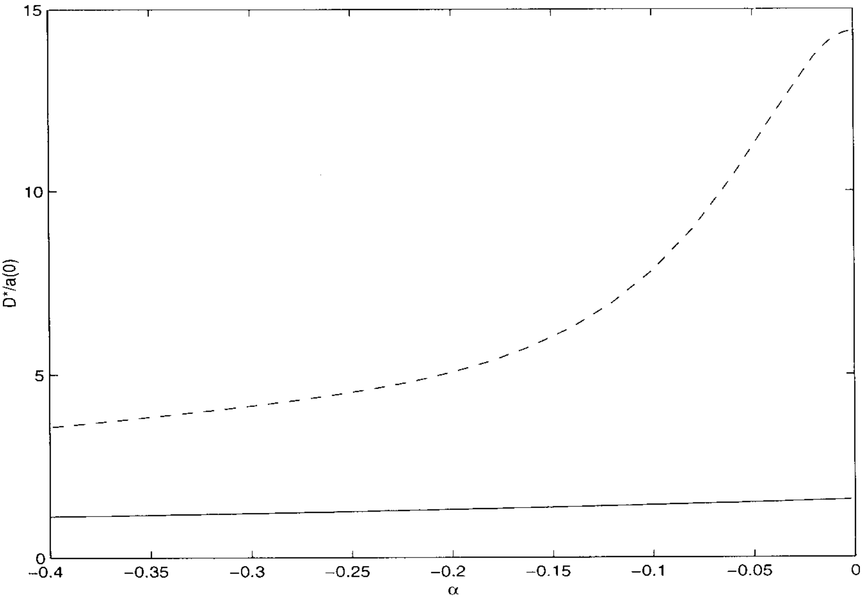

Fig. 4. The normalized effective diffusivities $D^*/a(0)$ versus $\alpha \in [-0.4, 0]$. The solid line corresponds to Eq. (18); the dashed line corresponds to Eq. (19).

here is that the effective diffusivity $D^*(D)$ is *nonperturbative* with respect to D in the sense that

$$\lim_{D \to 0} D^*(D) > D^*(0)$$

As shown in Fig. 4, $\lim_{D \to 0} D^*(D)/a(0)$ is nearly 10 times as large as the $D^*(0)/a(0)$ for α near zero. This is the first such example that we are aware of to demonstrate rigorously a nonperturbative effect of molecular diffusion on turbulent diffusion coefficient.

2. $D = 0$

Let $W_0(dk)$, $W_1(dk)$ be two independent, identically distributed Gaussian white-noise field in $k \in R$. Then the following Gaussian random fields

$$W_0(t, x, dk) := \cos(t\, |k|^{2\beta} + kx)\; W_0(dk) + \sin(t\, |k|^{2\beta} + kx)\; W_1(dk) \tag{20}$$

$$W_1(t, x, dk) := -\sin(t\, |k|^{2\beta} + kx)\; W_0(dk) + \cos(t\, |k|^{2\beta} + kx)\; W_1(dk) \tag{21}$$

are also independent and identically distributed. We have the spectral representation for V (see, e.g., ref. 24):

$$V(t, x) = \int_R \frac{a^{1/2}(|k|)}{|k|^{\alpha - 1/2}} W_0(t, x, dk) \tag{22}$$

In the absence of molecular diffusion, we have

$$y_{\varepsilon, \delta}(t) = \varepsilon \int_0^{t/\varepsilon^{2\delta}} V(s, 0)\, ds \tag{23}$$

and

$$\begin{aligned} \mathbf{E}(y_{\varepsilon, \delta}(t))^2 &= \varepsilon^2 \int_0^{t/\varepsilon^{2\delta}} \int_0^{t/\varepsilon^{2\delta}} R(s - s', 0)\, ds'\, ds \\ &= 4\varepsilon^2 \int_0^\infty \frac{a(|k|)}{|k|^{2\alpha + 4\beta - 1}} (1 - \cos(t\, |k|^{2\beta}\, \varepsilon^{-2\delta}))\, dk \end{aligned} \tag{24}$$

For $\alpha + 2\beta < 1$, the above integral is convergent, so $\mathbf{E}(y_{\varepsilon, \delta}(t))^2 = O(\varepsilon^2)$ for all $\delta > 0$. That is,

$$\mathbf{E}(y(t/\varepsilon^{2\delta}))^2 = O(1) \qquad \text{uniform in} \quad \varepsilon > 0,\ \delta > 0,\ t > 0 \tag{25}$$

This is what we call a *recycling motion.* For $\alpha + 2\beta > 1$, the integral becomes, after the change of variable $k = \xi \varepsilon^{\delta/\beta}$,

$$\mathbf{E}(y_{\varepsilon, \delta}(t))^2 = 4\varepsilon^2 \varepsilon^{-2\delta(\alpha + 2\beta - 1)/\beta} \int_0^\infty \frac{a(\xi \varepsilon^{\delta/\beta})}{\xi^{2\alpha + 4\beta - 1}} (1 - \cos(t\xi^{2\beta}))\, d\xi \tag{26}$$

Note that the integral is convergent at $\xi = \infty$ for $\alpha + 2\beta > 1$ and at $\xi = 0$ for $\alpha < 1$. Thus we have, by choosing

$$\delta = \frac{\beta}{\alpha + 2\beta - 1}$$

that

$$\mathbf{E}(y_{\varepsilon, \delta}(t))^2 \sim 4a(0) \int_0^\infty 4a(0) \int_0^\infty k^{1 - 2\alpha - 4\beta}(1 - \cos(k^{2\beta}))\, dk\; t^{(\alpha + 2\beta - 1)/\beta} \tag{27}$$

after another change of variable $\xi = kt^{-1/(2\beta)}$. Since, by definition (23), $y_{\varepsilon,\delta}(t)$ is a Gaussian process with stationary increments, the calculation (27) is sufficient for us to conclude

Theorem 1. Suppose $D = 0$. Then, for $\alpha + 2\beta > 1$, $y_{\varepsilon,\delta}(t)$ with

$$\delta = \frac{\beta}{\alpha + 2\beta - 1}$$

converges weakly to a fractional Brownian motion with the Hurst exponent $H = 1/(2\delta)$ and the variance $2D^*$ at time 1 given by

$$2D^* = 4a(0) \int_0^\infty k^{1-2\alpha-4\beta}(1 - \cos(k^{2\beta}))\, dk \tag{28}$$

In particular, the limiting fractional Brownian motion is persistent for $\alpha + \beta > 1$ and is antipersistent for $\alpha + \beta \leqslant 1$. For $\alpha + 2\beta < 1$, the limit is a fractional Brownian motion with $H = 0$ for any $\delta > 0$.

3. EFFECT OF INFRARED CUTOFF

Since the fractional Brownian motion limit is often the result of low wave numbers, it is natural to consider the effect of an infrared cutoff which is taken to be ε^γ, $\gamma > 0$. For simplicity of discussion, we assume that $a(k)$ is the characteristic function of the interval $[\varepsilon^\gamma, K_0]$ times a constant $a(0)$ where ε^γ, $\gamma > 0$, describes an (vanishing) infrared cutoff. By tracing the calculation in the previous section, it is easy to see that for $\alpha + 2\beta < 1$, $y_{\varepsilon,\delta}(t)$ converges weakly to the FBM with $H = 0$ for all $\gamma > 0$. For $\alpha + 2\beta \geqslant 1$, however, the answer is different. Equation (26) now becomes

$$\mathbf{E}(y_{\varepsilon,\delta}(t))^2 = 4a(0)\, \varepsilon^2 \varepsilon^{2\delta\beta/(1-\alpha-2\beta)} \int_{\varepsilon^{\gamma-\delta/\beta}}^{K_0 \varepsilon^{-\delta/\beta}} \xi^{1-2\alpha-4\beta}(1 - \cos(t\xi^{2\beta}))\, d\xi \tag{29}$$

If

$$\gamma > \frac{1}{\alpha + 2\beta - 1} \tag{30}$$

then, for $\delta = \beta/(\alpha + 2\beta - 1)$, we have $\gamma - \delta/\beta > 0$ and the rest of calculation in the previous section follows through and Theorem 1 remains valid. If Eq. (30) is violated, then the scaling limit may not exist! In summary, Theorem 1 holds for any infrared cutoff ε^γ in the power spectrum of V with

$$\gamma > \max(0, (\alpha + 2\beta - 1)^{-1})$$

4. $D > 0$

First we need some preparation before stating the main results for this case. Let F be the canonical Wiener measure defined on the space $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ where $\mathcal{X} = C[0, +\infty)$ and $\mathcal{B}$ is its Borel-σ algebra. For any $w \in \mathcal{X}$ we set

$$r_w(k, s, t) = \int_0^t \int_0^s \cos(k(w(u) - w(u'))) \, du \, du' \qquad \forall t, s \geqslant 0 \tag{31}$$

Proposition 1. For F almost all $w \in \mathcal{X}$, r_w satisfies

$$\int_R r_w^2(k, s, t)(1 + k^2)^r \, dk < +\infty, \qquad \forall 0 \leqslant r < 3/2, \quad s, t \geqslant 0 \tag{32}$$

By Proposition 1, the random function

$$R_w(s, t) := a(0) \int_R r_w(k, s, t) \frac{dk}{|k|^{2\alpha - 1}}, \qquad s, t \geqslant 0 \tag{33}$$

is well defined for any $\alpha > 0$ since we can choose $r > 3/2 - 2\alpha$. Below we list a number of properties of R_w.

Proposition 2. (1) (Hölder continuity) $R_w(t, s)$, $t, s \geqslant 0$ is Hölder continuous and positive definite for F almost all $w \in \mathcal{X}$.

(2) (Self similarity) The law of $R_w(\lambda t, \lambda s)$ and that of $\lambda^{2H} R_w(t, s)$ are identical for any $\lambda > 0$.

(3) (Stationary increments in the mean) The law of

$$R_w(t + h, s + h) + R_w(h, h) - (R_w(t + h, h) + R(h, s + h)), \qquad s, t, h \geqslant 0 \tag{34}$$

is independent of h and the law of $R_w(t, t)$ is identical to that of $D^*(w) \, t^{\alpha + 1}$ where D^* is given by

$$D^*(w) = \begin{cases} C_\alpha(0) \, D^{\alpha - 1} \int_0^1 \left(\int_0^s |w(s')|^{2(\alpha - 1)} \, ds' \right) ds, & \text{when} \quad \alpha > \frac{1}{2} \\ C_\alpha a(0) \, D^{\alpha - 1} \int_R (\varphi(x, w) - \varphi(0, w)) \, |x|^{2(\alpha - 1)} \, dx & \text{when} \quad \alpha < \frac{1}{2} \\ C_\alpha a(0) \, D^{-1/2} \varphi(0, w), & \text{when} \quad \alpha = \frac{1}{2} \end{cases} \tag{35}$$

where C_α is a constant depending on α and the function $\varphi(\cdot, w)$ in (35) is Hölder continuous with any exponent less than 1, defined via its Fourier transform $\hat{\varphi}$

$$\hat{\varphi}(k, w) := \left| \int_0^1 e^{ikw(s)}\, ds \right|^2 \qquad \forall k \in R, \quad w \in \mathscr{X} \tag{36}$$

Propositions 1 and 2 are proved in Section 4.4. By Proposition 2 we can define, for F almost all $w \in \mathscr{X}$, a unique (random) Gaussian measure Q_w on $(\mathscr{X}, \mathscr{B}(\mathscr{X}))$ corresponding to the covariance function R_w. The canonical processes $y(t, \omega)$, $t \geqslant 0$, $\omega \in \mathscr{X}$ associated with the measures Q_w need not have stationary increments, although on average they do as stated in Part 3 of Proposition 2. Moreover, (random) mean square displacement

$$\int (y(t) - y(s))^2 \, Q_w(dy), \qquad \forall t, s \geqslant 0$$

has the same distribution as $D^*(w)(t-s)^{(1+\alpha)}$, resembling that of a fractional Brownian motion with the Hurst exponent $H = (1+\alpha)/2$ and $2D^*(w)$ as the variance at time 1. So we call it a *local, fractional Brownian motion (LFBM)*. Our main results are stated in the following theorem.

Theorem 2. (1) Let $\alpha + \beta < 1$, or $\alpha < 0$. Then $\delta = 1$ and the weak limit of (15) is a Brownian motion with the diffusion coefficient

$$D^* = D + D \int_R \frac{a(|k|)}{|k|^{2\alpha - 3} \, (D^2 k^4 + |k|^{4\beta})} \, dk \tag{37}$$

(2) Let $\alpha + \beta > 1$, $\beta < 1$ and $0 < \alpha < 1$. Then

$$\delta = \frac{\beta}{\alpha + 2\beta - 1} \tag{38}$$

and the limit of (15) is a persistent FBM with the variance at time 1

$$2D^* = 4a(0) \int_0^\infty k^{1 - 2\alpha - 4\beta} (1 - \cos(k^{2\beta})) \, dk \tag{39}$$

and the Hurst exponent $1/2 < H = 1/(2\delta) < 1$.

(3) Let $\alpha + \beta > 1$, $\beta > 1$ and $0 < \alpha < 1$. Then $\delta = 1/(\alpha + 1)$, the limit of (15) is the measure

$$Q(A) = \int Q_w(A)\, F(dw), \qquad A \in \mathscr{B}(\mathscr{X}) \tag{40}$$

where Q_w is the law of a local FBM defined above. The measure Q is *self similar* with the Hurst exponent $H = (\alpha + 1)/2$ and the associated process has stationary increments.

The canonical process associated with Q in (40) is called a *composite fractional Brownian motion* because it is a random mixture of LFBMs with different covariance functions but the same Hurst exponent. It is a H-sssi process in the terminology of ref. 26 with a well-defined exponent $H = (\alpha + 1)/2$. It should be noted, however, that composite FBMs are *not* Gaussian.

In the following proof, C and C' denote generic constants independent of the scaling factor ε and wave number k.

4.1. Proof of Part 1—The Homogenization Regime

For the proof of (1) we adopt a martingale technique (see ref. 16 for general reference and ref. 9 in the homogenization context).

We first define the non-stationary *corrector field* by the stochastic integral

$$\begin{aligned}\chi(t, x) := &\int_R \frac{a^{1/2}(|k|)\, |k|^{2\beta}}{(|k|^{4\beta} + D^2 k^4)\, |k|^{\alpha - 1/2}} \left[W_1(dk) - W_1(t, x, dk) \right] \\ &- \int_R \frac{a^{1/2}(|k|)\, D\, |k|^2}{(|k|^{4\beta} + D^2 k^4)\, |k|^{\alpha - 1/2}} \left[-W_0(dk) + W_0(t, x, dk) \right]\end{aligned} \tag{41}$$

The corrector field has a finite second moment

$$\mathbf{E}(\chi(t, x))^2 = \int_R \frac{2a(|k|\, [1 - \cos(t\, |k|^{2\beta} + kx)]\, dk}{(|k|^{4\beta} + D^2 k^4)\, |k|^{2\alpha - 1}} < +\infty \tag{42}$$

for $\alpha < 0$ or $\alpha + 2\beta < 2$, including the region defined by $\alpha < 0$ or $\alpha + \beta < 1$.

The corrector field is, however, non-stationary since the integral in (42) depends on x. Its gradient, $(\partial_t \chi, \partial_x \chi)$ is of mean zero and stationary with finite second mement

$$\mathbf{E}[\, |\partial_t \chi(t, x)|^2 + D^2\, |\partial_x \chi(t, x)|^2] = \int_R \frac{a(|k|)\, dk}{|k|^{2\alpha - 1}} < +\infty, \qquad \forall \alpha < 1$$

independent of x. The utility of the corrector field in (47) is related to the *corrector equation*

$$\partial_t \chi(t, x) + D\, \partial_x^2 \chi(t, x) = V(t, x) \tag{43}$$

for almost all realizations of V and χ. We need the following result.

Proposition 3. Let $\alpha < 0$ or $\alpha + \beta < 1$. Then,

$$\lim_{\varepsilon \downarrow 0} P\left[\sup_{C_K} \varepsilon \left|\chi\left(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\right)\right| \geqslant \eta\right] = 0, \qquad \forall K, \eta > 0 \tag{44}$$

Here $C_K := \{(t, x)\colon \sqrt{|t|}, |x| \leqslant K\}$.

Proof. it is straightforward to check that

$$\begin{aligned}
\sigma_\varepsilon^2 &:= \sup_{C_K} \varepsilon^2\, \mathbf{E}\left|\chi\left(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\right)\right|^2 \\
&= \varepsilon^2 \int_R \frac{a(|k|)[1 - \cos(t\varepsilon^{-2} k^{2\beta} + \varepsilon^{-1} xk)]}{[k^{4\beta} + D^2 k^4]\, k^{2\alpha - 1}}\, dk \\
&\leqslant C\varepsilon^{2\rho}, \qquad \text{with} \quad \rho = \begin{cases} \min[1, (1-\alpha-\beta)/\beta] & \text{for} \quad \beta \leqslant 1, \quad \alpha + \beta < 1 \\ -\alpha, & \text{for} \quad \alpha < 0, \quad \beta > 1 \end{cases}
\end{aligned} \tag{45}$$

as $\varepsilon \to 0$.

To finish the proof of the lemma we use the Borell–Fernique–Talagrand estimate (ref. 2, Theorem 5.2, p. 120) to estimate the extremal behavior of a Gaussian field from its mean-square behavior. First note that

$$\begin{aligned}
&\mathrm{d}_\varepsilon((t, x), (s, y)) \\
&\quad := \varepsilon \left\{\mathbf{E}\left|\chi\left(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\right) - \chi\left(\frac{s}{\varepsilon^2}, \frac{y}{\varepsilon}\right)\right|^2\right\}^{1/2} \\
&\quad = \varepsilon \left(\int_R \frac{a(|k|)[1 - \cos((t-s)\,\varepsilon^{-2} k^{2\beta} + (x-y)\,\varepsilon^{-1} k)]}{[k^{4\beta} + D^2 k^4]\, k^{2\alpha-1}}\, dk\right)^{1/2}
\end{aligned} \tag{46}$$

defines a family of metrics on the space C_K by the law of $\varepsilon\chi(t/\varepsilon^2, x/\varepsilon)$. The proof of (45) shows that the diameter of the space C_K with respect to the metric d_ε is less than or equal to $C\varepsilon^\rho$ for a certain $\rho > 0$. By (46), we also know

$$d_\varepsilon((t, x), (s, y)) \leqslant C[|x-y| + \varepsilon^{-1}|t-s|] \qquad \forall (t, x), (s, y) \in C_K$$

Thus, the minimal number $N_\varepsilon(r)$ of d-balls with radius r needed to cover C_K, satisfies $N_\varepsilon(r) \leqslant Cr^{-M}$, for some constants C, $M > 0$ independent of r and ε. Thus by ref. 2 (Theorem 5.2, p. 120) we have

$$P\left[\sup_{C_K} \varepsilon \left|\chi\left(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\right)\right| \geqslant \eta\right] \leqslant C\eta^{M+1} \exp\left\{-\frac{\eta^2}{8\sigma_\varepsilon^2}\right\}$$

which vanishes in view of (45). ∎

By (14), (43) and the Ito formula we obtain

$$\varepsilon\chi\left(\frac{t}{\varepsilon^2}, \sqrt{2D}\,\varepsilon w_1\left(\frac{t}{\varepsilon^2}\right)\right) - \varepsilon\chi(0, 0) + \varepsilon y\left(\frac{t}{\varepsilon^2}\right)$$
$$= \varepsilon\sqrt{2D}\int_0^{t/\varepsilon^2} \partial_x\chi(s, \sqrt{2D}\,w_1(s))\,dx_1(s) + \sqrt{2D}\,\varepsilon w_2\left(\frac{t}{\varepsilon^2}\right) \tag{47}$$

Proposition 3 implies that

$$\lim_{\varepsilon\downarrow 0} P\otimes Q\left[\sup_{0\leqslant t\leqslant T} \varepsilon\left|\chi\left(\frac{t}{\varepsilon^2}, \sqrt{2D}\,\varepsilon w_1\left(\frac{t}{\varepsilon^2}\right)\right)\right| > \eta\right] = 0, \qquad \forall T, \eta > 0$$

so that $\varepsilon y(t/\varepsilon^2)$ is approximately a martingale, given by the right hand side of (47), which has the quadratic variation

$$2D\varepsilon^2 \int_0^{t/\varepsilon^2} (\partial_x\chi)^2\,(s, \sqrt{2D}\,w_1(s))\,ds + 2Dt \tag{48}$$

By a standard result (see, e.g., ref. 9, Proposition 1) we know that the process $\partial_x(t, \sqrt{2D}\,w_1(t))$, is stationary and ergodic over the product probability space $(\Omega\otimes\Sigma, \mathscr{V}\otimes\mathscr{W}, P\otimes Q)$. Therefore, with probability one (48) tends to, as $\varepsilon\to 0$,

$$2Dt(1 + \mathbf{E}(\partial_x\chi)^2) = 2D^*t$$

with D^* given by (37). The martingale invariance principle[17] then implies that (47) tends weakly as $\varepsilon\downarrow 0$ to a Brownian motion with diffusion coefficient given by (37). This completes the proof of Part 1 of Theorem 2. ∎

4.2. Proof of Part 2

Tightness. Let δ be as stated in the theorem. We first show that for $H=1/(2\delta)$ and any $T>0$, $r\in(0,1)$

$$\lim_{\varepsilon\downarrow 0}\int_0^T\int_0^T \frac{\mathbf{ME}[\,y_{\varepsilon,\delta}(t)-y_{\varepsilon,\delta}(s)]^2}{|t-s|^{2H+r}}\,dt\,ds=C<+\infty,\quad \forall 0\leqslant s\leqslant t\leqslant T \tag{49}$$

Since $H>1/2$ throughout this regime, (49) and Corollary 2.1.4 of ref. 27 then imply that

$$\lim_{\varrho\downarrow 0}\limsup_{\varepsilon\downarrow 0} P\Big(\sup_{\substack{0\leqslant s\leqslant t\leqslant T\\ t-s<\varrho}} |y_{\varepsilon,\delta}(t)-y_{\varepsilon,\delta}(s)|\geqslant h\Big)=0,\qquad \forall h<0$$

and the tightness of the laws of $y_{\varepsilon,\delta}(t)$, $t\geqslant 0$ for $0<\varepsilon\leqslant 1$ in $\mathscr{X}$ follows from Theorem 1.3.2 of ref. 27.

Note that

$$\mathbf{ME}[\,y_{\varepsilon,\delta}(t)-y_{\varepsilon,\delta}(s)]^2=\mathbf{ME}y^2_{\varepsilon,\delta}(|t-s|) \tag{50}$$

Without loss of generality for our calculation we set $s=0$. We have

$$\begin{aligned}
\mathbf{ME}y^2_{\varepsilon,\delta}(t)&=\frac{2\varepsilon^2}{\sqrt{2\pi}}\int_0^{t/\varepsilon^{2\delta}}\int_0^s\int_R R(s',\sqrt{2Ds'}\,u)\,\mathrm{e}^{-u^2/2}ds\,ds'\,du\\
&=\frac{2\varepsilon^{2(1-\delta)}}{\sqrt{2\pi}}\int_0^t\int_0^{s/\varepsilon^{2\delta}}\int_R\int_R \frac{a(|k|)\cos(|k|^{2\beta}s')\cos(k\sqrt{2Ds'}\,u)}{|k|^{2\alpha-1}}\\
&\quad\times \mathrm{e}^{-u^2/2}\,ds\,ds'\,du\,dk\\
&=2\varepsilon^{2+2(1-\alpha-2\beta)\,\delta/\beta}\int_0^t\int_R \frac{a(\varepsilon^{\delta/\beta}\,|\xi|)}{(D^2\xi^4\varepsilon^{4(1/\beta-1)}+|\xi|^{4\beta})\,|\xi|^{2\alpha-1}}\\
&\quad\times\{D\xi^2\varepsilon^{2(1/\beta-1)}[1-\mathrm{e}^{-D\varepsilon^{2(1/\beta-1)}\xi^2 s}\cos(|\xi|^{2\beta}s)]\\
&\quad+|\xi|^{2\beta}\sin(|\xi|^{2\beta}s)\,\mathrm{e}^{-D\varepsilon^{2(1/\beta-1)}\xi^2 s}\}\;ds\,d\xi
\end{aligned} \tag{51}$$

Since $1/\beta-1>0$, (49) has the same asymptotic as

$$2\int_0^T\int_0^T\left[\int_0^{|t-s|}\int_R \frac{a(\varepsilon^{\delta/\beta}\,|\xi|)}{|\xi|^{4\beta}\,|\xi|^{2\alpha-1}}\,|\xi|^{2\beta}\sin(|\xi|^{2\beta}s')\,ds'\,d\xi\right]\frac{ds\,dt}{|t-s|^{2H+r}}$$

$$\sim 4a(0)\int_0^T\int_0^T |t-s|^{-r}\,ds\,dt\int_0^\infty \xi^{1-2\alpha-4\beta}(1-\cos(\xi^{2\beta}))\,d\xi$$

The corresponding result for $D=0$ remains therefore unchanged.

Convergence of Finite Dimensional Distributions. Since $\varepsilon w_2(t/\varepsilon^{2\delta})$ tends to zero for $\delta < 1$, it suffices to show that finite dimensional distributions of the process

$$Z_\varepsilon(t) := \varepsilon \int_0^{t/\varepsilon^{2\delta}} V(s, \sqrt{2D}\, w_1(s))\, ds, \qquad t \geqslant 0 \tag{52}$$

converge weakly to a Gaussian process Z_t, $t \geqslant 0$ with stationary increments, zero mean and the variance $2D^* t^{2H}$ at time t. Let us choose arbitrary $\xi_1, \ldots, \xi_n \in R$, $0 \leqslant t_1 \leqslant \cdots \leqslant t_n$ and set $Y := \xi_1 Z_\varepsilon(t_1) + \cdots + \xi_n Z_\varepsilon(t_n)$. The r.v. Y conditioned on F is Gaussian, thus

$$\mathbf{M}\mathbf{E} e^{iY} = \mathbf{M} \exp(-\tfrac{1}{2}\, \mathbf{E} Y^2) \tag{53}$$

The term $\mathbf{E} Y^2$ can be expressed as a sum of products $\xi_i \xi_j \mathbf{E}[Z_\varepsilon(t_i)\, Z_\varepsilon(t_j)]$. Now we need to prove the convergence

$$\mathbf{E}[Z_\varepsilon(t_i)\, Z_\varepsilon(t_j)] \to \frac{D^*}{2} [t_i^{2H} + t_j^{2H} - (t_j - t_i)^{2H}] \qquad \text{as} \quad \varepsilon \downarrow 0 \tag{54}$$

which corresponds to the covariance function of a FBM with the Hurst exponent H and the unittime variance $2D^*$. for this we observe that

$$\begin{aligned}
&\mathbf{E}[Z_\varepsilon(t_i)\, Z_\varepsilon(t_j)] \\
&\quad = \varepsilon^{2(1-2\delta)} \int_0^{t_i} \int_0^{t_j} R\left(\frac{s-s'}{\varepsilon^{2\delta}}, \sqrt{D}\left[w_1\left(\frac{s}{\varepsilon^{2\delta}}\right) - w_1\left(\frac{s'}{\varepsilon^{2\delta}}\right)\right]\right) ds\, ds' \\
&\quad = \varepsilon^{2+2(1-\alpha-2\beta)\,\delta/\beta} \int_0^{t_i} \int_0^{t_j} \int_R \frac{a(\varepsilon^{\delta/\beta}\, |k|)}{|k|^{2\alpha-1}} \\
&\qquad \times \cos(|k|^{2\beta}\,(s-s'))\cos(\sqrt{2D}\,\varepsilon^{\delta(1/\beta-1)}(w_1^\varepsilon(s) - w_1^\varepsilon(s')))\, ds\, ds'\, dk
\end{aligned}$$

with $w_1^\varepsilon(t) := \varepsilon^\delta w_1(t/\varepsilon^{2\delta})$. Passing to the limit $\varepsilon \downarrow 0$, we obtain

$$\begin{aligned}
\lim_{\varepsilon \to 0} \xi_i \xi_j\, \mathbf{E}[Z_\varepsilon(t_i)\, Z_\varepsilon(t_j)] &= a(0) \int_0^{t_i} \int_0^{t_j} \int_R \frac{\cos(|k|^{2\beta}\,(s-s'))}{|k|^{2\alpha-1}}\, ds\, ds'\, dk \\
&= \frac{D^*}{2} [t_i^{2H} + t_j^{2H} - (t_j - t_i)^{2H}]
\end{aligned} \tag{55}$$

with $H = (\alpha + 2\beta - 1)/(2\beta)$ and D^* given by (39).

4.3. Proof of Part 3—Composite FBM

Tightness. Using the same argument as in the relevant part of the proof of Part 2 it is sufficient to show that for any $T > 0$ there exists $C > 0$ such that for any $T > 0$ and $r \in (0, 1)$ and we have

$$\lim_{\varepsilon \downarrow 0} \int_0^T \int_0^T \frac{\mathbf{ME} y^2_{\varepsilon, \delta}(|t-s|)}{|t-s|^{2H+r}} \, ds \, dt = C < +\infty \tag{56}$$

for $H > 1/2$. Repeating the relevant calculations from Part 2 we find

$$\mathbf{ME} y^2_{\varepsilon, \delta}(t) = 2\varepsilon^{2[1-(1+\alpha)\delta]} \int_0^t \int_R \frac{a(\varepsilon^\delta |\xi|) \, d\xi}{[D^2 \xi^4 + \varepsilon^{4(\beta-1)\delta} |\xi|^{4\beta}] \, |\xi|^{2\alpha-1}}$$
$$\times \{ D\xi^2 [1 - \mathrm{e}^{-D\xi^2 s} \cos(\varepsilon^{2(\beta-1)\delta} |\xi|^{2\beta} s)]$$
$$+ \varepsilon^{2(\beta-1)\delta} |\xi|^{2\beta} \sin(\varepsilon^{2(\beta-1)\delta} |\xi|^{2\beta} s) \, \mathrm{e}^{-D\xi^2 s} \} \, ds \, d\xi \tag{57}$$

Using $\delta = 1/(\alpha+1)$, $\beta > 1$, and the Lebesgue dominated convergence theorem, we obtain in the limit $\varepsilon \to 0$ that

$$\lim_{\varepsilon \to 0} \int_0^T \int_0^T \frac{\mathbf{ME} y^2_{\varepsilon, \delta}(|t-s|)}{|t-s|^{2H+r}} \, ds \, dt$$
$$= 2a(0) \, D^{-1+\alpha} \int_0^T \int_0^T |t-s|^{-r} \, ds \, dt \int_R (1 - k^{-2}(1 - \mathrm{e}^{-k^2})) \frac{dk}{|k|^{2\alpha+1}} \tag{58}$$

Note that (58) is a convergent integral for $0 < \alpha < 1$. In view of (58), we take $H = (1+\alpha)/2$.

Convergence of Finite Dimensional Distributions. By the invariance of a Brownian motion under the scaling $\varepsilon^\delta w_1(t/\varepsilon^{2\delta})$ we have that

$$\mathbf{ME} f_1(y_{\varepsilon, \delta}(t_1)) \cdots f_n(y_{\varepsilon, \delta}(t_n))$$
$$= \int F(dw) \left(\int \cdots \int \hat{f}_1(\xi_1) \cdots \hat{f}_n(\xi_n) \right.$$
$$\left. \times \exp\left\{ -\tfrac{1}{2} \sum_{i,j=1}^n R_{\varepsilon, w}(t_j, t_i) \, \xi_i \xi_j \right\} d\xi_1 \cdots d\xi_n \right)$$

where $\hat{f}_1,..., \hat{f}_n$ are the Fourier transforms of any $f_1,..., f_n \in C_0^\infty(R)$ and

$$R_{\varepsilon, w}(s, t) := \varepsilon^2 \int_0^{s/\varepsilon^{2\delta}} \int_0^{t/\varepsilon^{2\delta}} \mathbf{E}[V(u, w(u))\, V(u', w(u'))]\, du\, du'$$

defined for any fixed $w \in \mathscr{X}$. We show

Proposition 4. Let $0 < \alpha$, $\beta > 1$. Then

$$\lim_{\varepsilon \downarrow 0} \int |R_{\varepsilon, w}(s, t) - R_w(s, t)|\, F(dw) = 0 \tag{59}$$

with R_w defined by (33).

Proof. The left hand side of (59) is less than or equal to

$$\int_R \frac{dk}{|k|^{2\alpha - 1}} \int \bigg| \int_0^t \int_0^s \cos(kw(u) - w(u'))) \\ \times [a(0) - a(\varepsilon^\delta |k|) \cos(\varepsilon^{2(\beta - 1)} |k|^{2\beta} (u - u'))] \bigg| F(dw) \tag{60}$$

As an integral on the k space, the above expression has an integrand that converges pointwise to 0 for $\beta > 1$. The integrand of the k integral in (60) can be bounded by $C |k|^{1 - 2\alpha}/(1 + k^2)$ which is integrable for $0 < \alpha < 1$. The proposition now follows by the Lebesgue dominated convergence theorem.

Self-similarity and stationarity of the canonical process associated with the limiting measure follow from Parts 2 and 3 of Proposition 2.

4.4. Proof of Propositions 1 and 2

Proof of Proposition 1. Notice that

$$r_w w(k, t, s) = \mathrm{Re} \int_0^t \int_0^s e^{ik(w(u) - w(u'))}\, du\, du'$$

and, hence,

$$r_w^2(k, t, s) \leqslant \left| \int_0^t e^{ikw(u)}\, du \right|^2 \left| \int_0^s e^{ikw(u)}\, du \right|^2 \tag{61}$$

We also have

$$\int \left| \int_0^t e^{ikw(u)}\, du \right|^4 F(dw) \leqslant CT^2 \left(\frac{t}{k^2+1} \right)^2 \tag{62}$$

From (61), (62) and Jensen's inequality, we know that

$$\int r_w^2(k, s, t)(1+k^2)^r\, F(dw)$$

is integrable in k for $0 \leqslant r < 3/2$ and, hence, the proposition. ▮

Proof of Part 1, Proposition 2. We have

$$\begin{aligned} &\int (R_w(s, t) - R_w(s, t'))^{2q}\, F(dw) \\ &\quad \leqslant C \int \left(\int_{|k| \leqslant 1} \frac{(r_w(k, s, t) - r_w(k, s, t'))}{|k|^{2\alpha - 1}}\, dk \right)^{2q} F(dw) \\ &\qquad + C \int \left(\int_{|k| > 1} \frac{(r_w(k, s, t) - r_w(k, s, t'))}{|k|^{2\alpha - 1}}\, dk \right)^{2q} F(dw) \end{aligned} \tag{63}$$

Since $|r_w(k, t, s) - r_w(k, t', s)| \leqslant |t - t'|\, s$, the first integral can be bounded by

$$|t - t'|^{2q}\, s^{2q} \left(\int_{|k| \leqslant 1} \frac{dk}{|k|^{2\alpha - 1}} \right)^{2q}$$

which is finite for $\alpha < 1$. Using the Jensen inequality for the integral with respect to the finite measure $1_{|k| \geqslant 1}\, dk/|k|^{2\alpha+1}$, the second term can be bounded by

$$\begin{aligned} &C \int_{|k| > 1} \frac{|k|^{4q}}{|k|^{2\alpha+1}} \left(\int |r_w(k, s, t) - r_w(k, s, t')|^{2q}\, F(dw) \right) dk \\ &\quad \leqslant C' s^q\, |t - t'|^q \int_{|k| > 1} \frac{1}{|k|^{2\alpha+1}}\, dk \end{aligned}$$

which is finite for $\alpha > 0$.

Similarly, we can show that

$$\begin{aligned} &\int (R_w(s, t) - R_w(s', t'))^{2q}\, F(dw) \\ &\quad \leqslant C(|t - t'|^q + |s - s'|^q) \qquad \forall 0 \leqslant s \leqslant S, \quad 0 \leqslant t \leqslant T \end{aligned}$$

which guarantees the existence of Hölder continuous modification of R_w (see ref. 1, Theorem 3.2.5 and its corollary).

Positive definiteness of R_w is obvious.

Part 2, Proposition 2. From (31) we have $r_w(k, \lambda s, \lambda t)$ and $\lambda^2 r_w(k\sqrt{\lambda}, s, t)$ are identically distributed. Thus,

$$R_w(\lambda s, \lambda t) = a(0)\,\lambda^2 \int_R r_w(\sqrt{\lambda}\,k, s, t)\,\frac{dk}{|k|^{2\alpha-1}}$$

$$= a(0)\,\lambda^{\alpha+1} \int_R r_w(\xi, s, t)\,\frac{d\xi}{|\xi|^{2\alpha-1}} \tag{64}$$

after a change of variable $\xi = \sqrt{\lambda}\,k$.

Part 3, Proposition 2. By (31) and (33), (34) can be written as

$$\int_h^{t+h} \int_h^{s+h} \cos(k(w(u) - w(u')))\,du\,du'$$

whose law is independent of h since $w(u)$ has stationary increments.

We also know that $R_w(t, t)$ and $D^*(w)\,t^{2H}$ are identically distributed with

$$D^*(w) := R_w(1, 1)$$

$$= a(0) \int_R \frac{1}{|k|^{2\alpha-1}} \int_0^1 \int_0^1 \cos(kw(u) - w(u')))\,du\,du'\,dk$$

$$= a(0) \int_R \frac{\hat{\varphi}(k, w)\,dk}{|k|^{2\alpha-1}} \tag{65}$$

where $\hat{\varphi}$ is given by (36). By Proposition 1 the function φ belongs to the Sobolev space $H^r(R)$, $\forall r < 3/2$. The classical Sobolev embedding theorem[15] then implies that φ is Hölder continuous with any exponent less than 1.

To simplify (65) we note that the inverse Fourier transform of $1/|k|^{2\alpha-1}$ is the distribution

$$F(f) := \begin{cases} C_\alpha a(0) \int_R |x|^{2(\alpha-1)} f(x)\,dx, & \alpha > \frac{1}{2} \\ C_\alpha a(0)\, f(0), & \alpha = \frac{1}{2} \\ C_\alpha a(0) \int_R \frac{f(x) - f(0)}{|x|^{2(1-\alpha)}}\,dx, & \alpha < \frac{1}{2} \end{cases}$$

for any Schwartz function f. We further observe that this distribution can actually be extended to test functions such as φ. Indeed, we have

$$|F(f)| = \left| \int_R \frac{\hat{f}(k)\,dk}{|k|^{2\alpha-1}} \right|$$

$$\leqslant \|\hat{f}\|_{L^\infty} \int_{|k|\leqslant 1} \frac{dk}{|k|^{2\alpha-1}} + \|f\|_{H^r(R)} \left(\int_{|k|\geqslant 1} \frac{dk}{|k|^{2\alpha-1}\,|k|^{2r}} \right)^{1/2} < \infty$$

for any $r > 1$ and $0 < \alpha < 1$. So the proof is now complete.

5. ASYMPTOTICS AS $D \to 0$

In the section we analyze the asymptotics of the effective diffusivity

$$D^* = D + D \int_R \frac{a(k)}{k^{2\alpha-3}(D^2k^4 + k^{4\beta})}\,dk \tag{66}$$

as D tends to zero, which is valid in the Brownian motion regime ($\alpha + \beta < 1$ or $\alpha < 0$). For $\alpha + 2\beta < 2$, (66) clearly has the asymptotics

$$D^* \sim D\left(1 + \int_R \frac{a(|k|)}{|k|^{2\alpha+4\beta-3}}\,dk\right) \tag{67}$$

because the integral is convergent. For $\alpha + 2\beta > 2$, the integral in (67) is divergent, therefore the asymptotics of D^* is more subtle. Note that $\alpha < 0$, $\beta > 1$ in this part of the Brownian motion regime. Let

$$k^* = D^{1/2(\beta-1)}$$

so that $D^2(k^*)^4 = (k^*)^{4\beta}$. It is easy to see that

$$D \int_{|k| \ll k^*} \frac{a(|k|)}{|k|^{2\alpha-3}\, D^2k^4}\,dk \ll \frac{a(0)}{-\alpha} D^{(1-\alpha-\beta)/(\beta-1)}$$

This shows that the effective diffusivity D^* mainly comes from the wave numbers of order k_* as D tends to zero. So after making the change of variable $\xi := k/k^*$, we obtain

$$D^* \sim D + D^{(1-\alpha-\beta)/(\beta-1)} 2a(0) \int_0^\infty \frac{d\xi}{\xi^{2\alpha-3}(\xi^4 + \xi^{4\beta})}, \qquad \text{for} \quad \beta > 1 \tag{68}$$

which is finite for $\alpha+2\beta>2$, $\alpha<0$. Moreover, the integral in (68) dominates over D in the region.

The asymptotics demonstrate the phenomenon of convection enhanced diffusion $D^* \gg D$ as

$$p := (1-\alpha-\beta)/(\beta-1) < 1, \qquad \text{for} \quad \alpha+2\beta>2$$

Moreover, the exponent becomes negative

$$-1 < p = (1-\alpha-\beta)/(\beta-1) < 0, \qquad \text{for} \quad \alpha+\beta>1$$

and results in a divergent effective diffusivity as D tends to zero. The range $p \in [-1, 1]$ is the fullest possible in any case and is fully realized in this regime.

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